Explicit Formulas for the Sequence of Outer Vecten Triangles

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In 1817, M. Vecten, a French Mathematician who taught Mathematics with Gergonne at the University of Nîmes in France, published a study of the figure of three squares erected on the sides of a triangle.

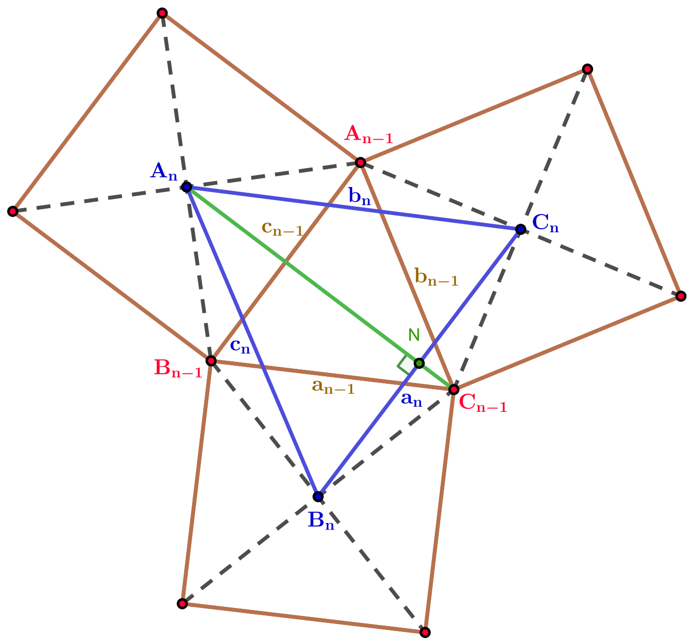

fig.1, Outer Vecten Triangle

When the squares lie outside the triangle, the triangle with vertices the centers of the squares is called outer Vecten triangle, (fig. 1), [**1**,**2**]. Given a triangle, one can construct a sequence $\{T_n\}_{n=0}^{\infty}$ of outer Vecten triangles with T_0 the given triangle and T_n the outer Vecten triangle of the T_{n-1} triangle.

In figure 1, we have constructed the terms $T_{n-1} \equiv A_{n-1}B_{n-1}C_{n-1}$, $T_n \equiv A_nB_nC_n$ of such a sequence, assuming as $T_0 \equiv A_0B_0C_0$ a given initial triangle with sidelengths a_0, b_0, c_0 and area $(A_0B_0C_0)$.

The sidelengths a_n, b_n, c_n and area $(A_nB_nC_n)$ of the n-th term of the sequence $\{T_n\}_{n=0}^{\infty}$ satisfy the following implicit relations, [**1**]

$$a_n^2 = \frac{a_{n-1}^2 + b_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2}, \tag{1}$$

$$b_n^2 = \frac{b_{n-1}^2 + c_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2}, \tag{2}$$

$$c_n^2 = \frac{c_{n-1}^2 + a_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2}, \tag{3}$$

$$8\,(A_nB_nC_n) = a_{n-1}^2 + b_{n-1}^2 + c_{n-1}^2 + 8\,(A_{n-1}B_{n-1}C_{n-1}). \tag{4}$$

Notice that in [**1**] the relations (1), (2), (3) are stated without the factor 4 multiplying the area $(A_{n-1}B_{n-1}C_{n-1})$, probably by same kind of mistake. As we shall prove the factor 4 should appear in the relations.

Our goal though is not to correct a misprint or the overlooking of some multiplicative factor but to present explicit formulas for the determination of the sidelengths a_n, b_n, c_n and area $(A_nB_nC_n)$ of the n-th term of the sequence $\{T_n\}_{n=0}^{\infty}$ without having to compute the sidelengths and areas of all the intermediate terms.

More precisely we shall prove that if $A_0B_0C_0$ is a given initial triangle with sidelengths a_0, b_0, c_0 and area $(A_0B_0C_0)$, then the sidelengths a_n, b_n, c_n and area $(A_nB_nC_n)$ of the n-th term of the sequence $\{T_n\}_{n=0}^{\infty}$ of outer Vecten triangles are given explicitly by the formulas,

$$a_n^2 = \ell_1\left(\frac{1+\sqrt{3}\,i}{4}\right)^n + \ell_2\left(\frac{1-\sqrt{3}\,i}{4}\right)^n + \ell_3\left(\frac{2+\sqrt{3}}{2}\right)^n + \ell_4\left(\frac{2-\sqrt{3}}{2}\right)^n, \quad (5)$$

with ℓ_i, $i = 1, 2, 3, 4$ given explicitly by,

$$\begin{aligned}
\ell_1 &= \frac{1}{3}\,a_0^2 + \frac{-1-\sqrt{3}\,i}{6}\,b_0^2 + \frac{-1+\sqrt{3}\,i}{6}\,c_0^2,\\
\ell_2 &= \frac{1}{3}\,a_0^2 + \frac{-1+\sqrt{3}\,i}{6}\,b_0^2 + \frac{-1-\sqrt{3}\,i}{6}\,c_0^2,\\
\ell_3 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 + \frac{2}{3}\,(A_0B_0C_0),\\
\ell_4 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 - \frac{2}{3}\,(A_0B_0C_0),
\end{aligned}$$

$$b_n^2 = d_1\left(\frac{1+\sqrt{3}\,i}{4}\right)^n + d_2\left(\frac{1-\sqrt{3}\,i}{4}\right)^n + d_3\left(\frac{2+\sqrt{3}}{2}\right)^n + d_4\left(\frac{2-\sqrt{3}}{2}\right)^n, \quad (6)$$

with d_i, $i = 1, 2, 3, 4$ given explicitly by,

$$\begin{aligned}
d_1 &= \frac{-1+\sqrt{3}\,i}{6}\,a_0^2 + \frac{1}{3}\,b_0^2 + \frac{-1-\sqrt{3}\,i}{6}\,c_0^2,\\
d_2 &= \frac{-1-\sqrt{3}\,i}{6}\,a_0^2 + \frac{1}{3}\,b_0^2 + \frac{-1+\sqrt{3}\,i}{6}\,c_0^2,\\
d_3 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 + \frac{2}{3}\,(A_0B_0C_0),\\
d_4 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 - \frac{2}{3}\,(A_0B_0C_0),
\end{aligned}$$

$$c_n^2 = e_1\left(\frac{1+\sqrt{3}\,i}{4}\right)^n + e_2\left(\frac{1-\sqrt{3}\,i}{4}\right)^n + e_3\left(\frac{2+\sqrt{3}}{2}\right)^n + e_4\left(\frac{2-\sqrt{3}}{2}\right)^n, \quad (7)$$

with e_i, $i = 1, 2, 3, 4$ given explicitly by,

$$\begin{aligned} e_1 &= \frac{-1-\sqrt{3}\,i}{6}\,a_0^2 + \frac{-1+\sqrt{3}\,i}{6}\,b_0^2 + \frac{1}{3}\,c_0^2, \\ e_2 &= \frac{-1+\sqrt{3}\,i}{6}\,a_0^2 + \frac{-1-\sqrt{3}\,i}{6}\,b_0^2 + \frac{1}{3}\,c_0^2, \\ e_3 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 + \frac{2}{3}\,(A_0B_0C_0), \\ e_4 &= \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 - \frac{2}{3}\,(A_0B_0C_0). \end{aligned}$$

And for the area,

$$(A_nB_nC_n) = \frac{\sqrt{3}\,K_1}{48}\left(\frac{2+\sqrt{3}}{2}\right)^n - \frac{\sqrt{3}\,K_2}{48}\left(\frac{2-\sqrt{3}}{2}\right)^n + \frac{1}{2}\,(A_0B_0C_0), \quad (8)$$

with K_1, K_2 given explicitly by,

$$\begin{aligned} K_1 &= a_0^2 + b_0^2 + c_0^2 + 4\sqrt{3}\,(A_0B_0C_0), \\ K_2 &= a_0^2 + b_0^2 + c_0^2 - 4\sqrt{3}\,(A_0B_0C_0). \end{aligned}$$

Example. Let the given initial triangle $A_0B_0C_0$ have $a_0 = 3$, $b_0 = 5$, $c_0 = 6$. Then, $(A_0B_0C_0) = 2\sqrt{14}$ and the sidelengths a_n, b_n, c_n of the terms of the sequence $\{T_n\}_{n=0}^{\infty}$ of outer Vecten triangles are given by the explicit formulas (5), (6), (7) with,

$$\begin{aligned} \ell_1 &= \frac{-43+11\sqrt{3}\,i}{6}, \ \ell_2 = \frac{-43-11\sqrt{3}\,i}{6}, \ \ell_3 = \frac{70+8\sqrt{14}}{6}, \\ \ell_4 &= \frac{70-8\sqrt{14}}{6}, \\ d_1 &= \frac{5-27\sqrt{3}\,i}{6}, \ d_2 = \frac{5+27\sqrt{3}\,i}{6}, \ d_3 = \frac{70+8\sqrt{14}}{6}, \\ d_4 &= \frac{70-8\sqrt{14}}{6}, \\ e_1 &= \frac{38+16\sqrt{3}\,i}{6}, \ e_2 = \frac{38-16\sqrt{3}\,i}{6}, \ e_3 = \frac{70+8\sqrt{14}}{6}, \\ e_4 &= \frac{70-8\sqrt{14}}{6}. \end{aligned}$$

The area $(A_nB_nC_n)$ of the terms of the sequence $\{T_n\}_{n=0}^{\infty}$ of outer Vecten triangles is given by (8) with, $K_1 = 70 + 8\sqrt{42}$, $K_2 = 70 - 8\sqrt{42}$.

The Explicit Formulas for the Sidelengths

In this section we shall present the determination of the explicit formulas (5), (6), (7). As an intermediate step we shall provide the proof of the correct implicit relations (1), (2), (3). All geometric object in the following proofs refer to figure 1.

Applying the law of cosines in triangle $B_nC_{n-1}C_n$, we take,

$$\begin{aligned} a_n^2 &= \frac{a_{n-1}^2}{2} + \frac{b_{n-1}^2}{2} - 2\,\frac{\sqrt{2}\,a_{n-1}}{2}\,\frac{\sqrt{2}\,b_{n-1}}{2}\,\cos\left(\frac{\pi}{2} + \hat{C}_{n-1}\right) = \\ &= \frac{a_{n-1}^2 + b_{n-1}^2 + 2\,a_{n-1}\,b_{n-1}\,\sin(\hat{C}_{n-1})}{2} = \\ &= \frac{a_{n-1}^2 + b_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2}\,. \end{aligned}$$

Applying the law of cosines in triangles $C_nA_{n-1}A_n$, $A_nB_{n-1}B_n$ and following similar reasoning as above we take (2), (3). Theorem 4.81, [**3, page 96**] implies that,

$$A_nC_{n-1} = B_nC_n = a_n \text{ and } A_nC_{n-1} \text{ is perpendicular to } B_nC_n = a_n, \tag{9}$$

$$B_n\widehat{A_nC}_{n-1} + A_n\widehat{B_nC}_n = 90^\circ. \tag{10}$$

By the law of cosines in triangle $A_nB_nC_{n-1}$ we take a proof of (4) namely,

$$\begin{aligned} \frac{a_{n-1}^2}{2} &\overset{(9)}{=} a_n^2 + c_n^2 - 2\,a_n\,c_n\,\cos(B_n\widehat{A_nC}_{n-1}) \overset{(10)}{=} \\ &= a_n^2 + c_n^2 - 2\,a_n\,c_n\,\sin(A_n\widehat{B_nC}_n) \overset{(1,3)}{=} \\ &= \frac{2\,a_{n-1}^2 + b_{n-1}^2 + c_{n-1}^2 + 8\,(A_{n-1}B_{n-1}C_{n-1})}{2} - \\ &\quad -4\,(A_nB_nC_n) \Rightarrow \\ 8\,(A_nB_nC_n) &= a_{n-1}^2 + b_{n-1}^2 + c_{n-1}^2 + 8\,(A_{n-1}B_{n-1}C_{n-1}). \end{aligned}$$

So far we have proved the implicit relations (1), (2), (3), (4). Now we proceed to find the linear recurrence relation satisfied by the sidelengths of the terms of sequence $\{T_n\}_{n=0}^{\infty}$.

$$\begin{aligned} a_{n+1}^2 &\overset{(1)}{=} \frac{a_n^2 + b_n^2 + 4\,(A_nB_nC_n)}{2} \overset{(1,2)}{=} \\ &= \frac{a_{n-1}^2 + 2\,b_{n-1}^2 + c_{n-1}^2 + 8\,(A_{n-1}B_{n-1}C_{n-1}) + 8\,(A_nB_nC_n)}{4} \overset{(4)}{=} \\ &= \frac{2\,a_{n-1}^2 + 3\,b_{n-1}^2 + 2\,c_{n-1}^2 + 16\,(A_{n-1}B_{n-1}C_{n-1})}{4}\,, \end{aligned} \tag{11}$$

$$\begin{aligned} a_{n+1}^2 - a_n^2 &\overset{(1,11)}{=} \frac{b_{n-1}^2 + 2\,c_{n-1}^2 + 8\,(A_{n-1}B_{n-1}C_{n-1})}{4} = \\ &= \frac{1}{2}\,b_n^2 + \frac{1}{2}\,\frac{c_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2}\,. \end{aligned} \tag{12}$$

The implicit relations (2), (3) give the two alternative expressions,

$$\frac{c_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2} \overset{(2)}{=} b_n^2 - \frac{b_{n-1}^2}{2}\,, \tag{13}$$

$$\frac{c_{n-1}^2 + 4\,(A_{n-1}B_{n-1}C_{n-1})}{2} \overset{(3)}{=} c_n^2 - \frac{a_{n-1}^2}{2}\,. \tag{14}$$

(13), (14) in turn produce two alternative expressions for (12) namely,

$$a_{n+1}^2 - a_n^2 \overset{(13)}{=} \frac{1}{2}\,b_n^2 + \frac{1}{2}\left(b_n^2 - \frac{b_{n-1}^2}{2}\right) = b_n^2 - \frac{b_{n-1}^2}{4}\,, \tag{15}$$

$$a_{n+1}^2 - a_n^2 \overset{(14)}{=} \frac{1}{2}\,b_n^2 + \frac{1}{2}\left(c_n^2 - \frac{a_{n-1}^2}{2}\right) = \frac{b_n^2 + c_n^2}{2} - \frac{a_{n-1}^2}{4}\,. \tag{16}$$

(15) implies,

$$a_n^2 + b_n^2 = a_{n+1}^2 + \frac{b_{n-1}^2}{4}\,. \tag{17}$$

(16) implies.

$$\frac{b_n^2 + c_n^2}{2} = a_{n+1}^2 - a_n^2 + \frac{a_{n-1}^2}{4} \text{ or, } \frac{a_n^2 + b_n^2 + c_n^2}{2} = a_{n+1}^2 - \frac{a_n^2}{2} + \frac{a_{n-1}^2}{4}\,. \tag{18}$$

Applying the same reasoning as above we take,

$$b_n^2 + c_n^2 = b_{n+1}^2 + \frac{c_{n-1}^2}{4}\,, \tag{19}$$

$$\frac{a_n^2 + b_n^2 + c_n^2}{2} = b_{n+1}^2 - \frac{b_n^2}{2} + \frac{b_{n-1}^2}{4}\,, \tag{20}$$

$$c_n^2 + a_n^2 = c_{n+1}^2 + \frac{a_{n-1}^2}{4}\,, \tag{21}$$

$$\frac{a_n^2 + b_n^2 + c_n^2}{2} = c_{n+1}^2 - \frac{c_n^2}{2} + \frac{c_{n-1}^2}{4}\,, \tag{22}$$

Adding by parts (17), (19), (21) the first key recurrence relation,

$$2\,(a_n^2 + b_n^2 + c_n^2) = a_{n+1}^2 + b_{n+1}^2 + c_{n+1}^2 + \frac{a_{n-1}^2 + b_{n-1}^2 + c_{n-1}^2}{4}\,, \tag{23}$$

is produced. The latter becomes functional when combined with (18),

$$a_n^2 + b_n^2 + c_n^2 = 2\,a_{n+1}^2 - a_n^2 + \frac{a_{n-1}^2}{2}\,,$$

implies the second key recurrence relation,

$$\begin{aligned} 16\,a_{n+2}^2 &= 40\,a_{n+1}^2 - 24\,a_n^2 + 10\,a_{n-1}^2 - a_{n-2}^2 \text{ or,} \\ 16\,a_n^2 &= 40\,a_{n-1}^2 - 24\,a_{n-2}^2 + 10\,a_{n-3}^2 - a_{n-4}^2\,, \end{aligned} \tag{24}$$

which is a linear recurrence relation of the fourth order. By exactly the same reasoning, (23) combined with (20) and (22) implies,

$$16\,b_n^2 = 40\,b_{n-1}^2 - 24\,b_{n-2}^2 + 10\,b_{n-3}^2 - b_{n-4}^2\,, \tag{25}$$

$$16\,c_n^2 = 40\,c_{n-1}^2 - 24\,c_{n-2}^2 + 10\,c_{n-3}^2 - c_{n-4}^2\,, \tag{26}$$

which are the linear recurrence relations satisfied by the sequences of the other two sidelengths. The characteristic equation of (24), (25), (26) is,

$$16\,x^4 - 40\,x^3 + 24\,x^2 - 10\,x + 1 = 0,$$

which factors to $(4\,x^2 - 2\,x + 1)\,(4\,x^2 - 8\,x + 1) = 0$ with roots,

$$x_1 = \frac{1+\sqrt{3}\,i}{4}\,,\; x_2 = \frac{1-\sqrt{3}\,i}{4}\,,\; x_3 = \frac{2+\sqrt{3}}{2}\,,\; x_4 = \frac{2-\sqrt{3}}{2}\,.$$

By the theory of linear recurrence relations, (5), (6), (7) are the explicit formulas expressing the sidelengths a_n, b_n, c_n of the n-th term of sequence $\{T_n\}_{n=0}^{\infty}$.

The constants ℓ_i, d_i, e_i, $i = 1, 2, 3, 4$ are produced as solutions of the three linear systems,

$$\begin{bmatrix} x_1^0 & x_2^0 & x_3^0 & x_4^0 \\ x_1^1 & x_2^1 & x_3^1 & x_4^1 \\ x_1^2 & x_2^2 & x_3^2 & x_4^2 \\ x_1^3 & x_2^3 & x_3^3 & x_4^3 \end{bmatrix} \begin{bmatrix} \ell_1 \text{ or, } d_1 \text{ or, } e_1 \\ \ell_2 \text{ or, } d_2 \text{ or, } e_2 \\ \ell_3 \text{ or, } d_3 \text{ or, } e_3 \\ \ell_4 \text{ or, } d_4 \text{ or, } e_4 \end{bmatrix} = \begin{bmatrix} a_0^2 \text{ or, } b_0^2 \text{ or, } c_0^2 \\ a_1^2 \text{ or, } b_1^2 \text{ or, } c_1^2 \\ a_2^2 \text{ or, } b_2^2 \text{ or, } c_2^2 \\ a_3^2 \text{ or, } b_3^2 \text{ or, } c_3^2 \end{bmatrix}.$$

The only sidelength terms of $\{T_n\}_{n=0}^{\infty}$ which we have to determine with respect to $a_0, b_0, c_0, (A_0B_0C_0)$ by the use of the implicit relations (1), (2), (3) are a_n, b_n, c_n for $n = 1, 2, 3$. All the other sidelength terms a_n, b_n, c_n for $n \in \mathbb{N} - \{1, 2, 3\}$ are given in a straightforward way by (5), (6), (7).

The Explicit Formula for the Areas

The linear recurrence relation (23) is the key for producing the explicit formula (8). (23) is a linear recurrence relation of the second order with characteristic equation,

$$y^2 - 2\,y + \frac{1}{4} = 0,$$

with roots $y_1 = \frac{2+\sqrt{3}}{2} = x_3$, $y_2 = \frac{2-\sqrt{3}}{2} = x_4$. The latter implies that,

$$a_n^2 + b_n^2 + c_n^2 = k_1 \left(\frac{2+\sqrt{3}}{2}\right)^n + k_2 \left(\frac{2-\sqrt{3}}{2}\right)^n, \tag{27}$$

with k_1, k_2 constants depending on $a_0^2 + b_0^2 + c_0^2$, $a_1^2 + b_1^2 + c_1^2$. Since,

$$a_1^2 + b_1^2 + c_1^2 \overset{(1,2,3)}{=\!=} a_0^2 + b_0^2 + c_0^2 + 6\,(A_0B_0C_0),$$

it is straight forward to solve the linear two by two system produced by (27) for $n = 0, 1$ to take,

$$k_1 = \frac{1}{2}\,(a_0^2 + b_0^2 + c_0^2 + 4\sqrt{3}\,(A_0B_0C_0)), \tag{28}$$

$$k_2 = \frac{1}{2}\,(a_0^2 + b_0^2 + c_0^2 - 4\sqrt{3}\,(A_0B_0C_0)), \tag{29}$$

(4) and (27) combined together imply,

$$8\,(A_nB_nC_n) = k_1\left(\frac{2+\sqrt{3}}{2}\right)^{n-1} + k_2\left(\frac{2-\sqrt{3}}{2}\right)^{n-1} + 8\,(A_{n-1}B_{n-1}C_{n-1}). \tag{30}$$

Adding by parts relations (30) for $n = 1, 2, \ldots, m$ we take,

$$8\,(A_mB_mC_m) = k_1\sum_{n=1}^{m}\left(\frac{2+\sqrt{3}}{2}\right)^{n-1} + k_2\sum_{n=1}^{m}\left(\frac{2-\sqrt{3}}{2}\right)^{n-1} + 8\,(A_0B_0C_0) =$$

$$= \frac{k_1}{\sqrt{3}}\left[\left(\frac{2+\sqrt{3}}{2}\right)^{m} - 1\right] - \frac{k_2}{\sqrt{3}}\left[\left(\frac{2-\sqrt{3}}{2}\right)^{m} - 1\right] + $$
$$+8\,(A_0B_0C_0) \Rightarrow$$

$$(A_mB_mC_m) = \frac{\sqrt{3}\,k_1}{24}\left(\frac{2+\sqrt{3}}{2}\right)^{m} - \frac{\sqrt{3}\,k_2}{24}\left(\frac{2-\sqrt{3}}{2}\right)^{m} +$$
$$+\frac{\sqrt{3}\,(k_2 - k_1)}{24} + (A_0B_0C_0),$$

and (8) follows from (28), (29) and $m = n$.

An Application of the Explicit Formulas

In problem proposal 1130, [**4**] the reader is asked to prove that if an initial triangle $A_0B_0C_0$ is given and $\{T_n\}_{n=0}^{\infty}$ is the sequence of outer Vecten triangles with respect to $A_0B_0C_0$, then a positive real constant γ exists such that,

$$\lim_{n\to\infty}\frac{a_n}{\gamma^n} = \lim_{n\to\infty}\frac{b_n}{\gamma^n} = \lim_{n\to\infty}\frac{c_n}{\gamma^n}\,, \tag{31}$$

is a positive real number. The latter means that if the terms of $\{T_n\}_{n=0}^{\infty}$ are scaled by $1/\gamma^n$, then the sequence $\{T_n/\gamma^n\}_{n=0}^{\infty}$ converges to an equilatelar triangle.

Taking $\gamma = \sqrt{x_3} = \sqrt{\frac{2+\sqrt{3}}{2}}$ and dividing both parts of (5), (6), (7) by $\gamma^{2\,n}$ we have,

$$\lim_{n\to\infty}\frac{a_n^2}{\gamma^{2\,n}} = \lim_{n\to\infty}\frac{b_n^2}{\gamma^{2\,n}} = \lim_{n\to\infty}\frac{c_n^2}{\gamma^{2\,n}} = \frac{1}{6}\,a_0^2 + \frac{1}{6}\,b_0^2 + \frac{1}{6}\,c_0^2 + \frac{2}{3}\,(A_0B_0C_0), \tag{32}$$

since,

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{x_1}{\gamma^2}\right)^n &= \lim_{n\to\infty}\left[\frac{\cos(n\,\pi/3)}{(2+\sqrt{3})^n}+\frac{\sin(n\,\pi/3)}{(2+\sqrt{3})^n}\,i\right]=0,\\
\lim_{n\to\infty}\left(\frac{x_2}{\gamma^2}\right)^n &= \lim_{n\to\infty}\left[\frac{\cos(n\,5\,\pi/3)}{(2+\sqrt{3})^n}+\frac{\sin(n\,5\,\pi/3)}{(2+\sqrt{3})^n}\,i\right]=0,\\
\lim_{n\to\infty}\left(\frac{x_3}{\gamma^2}\right)^n &= \lim_{n\to\infty}\left(\frac{\gamma^2}{\gamma^2}\right)^n=1,\\
\lim_{n\to\infty}\left(\frac{x_4}{\gamma^2}\right)^n &= \lim_{n\to\infty}\left(\frac{1}{7+4\sqrt{3}}\right)^n=0,
\end{aligned}$$

with,

$$x_1=\frac{1+\sqrt{3}\,i}{4}\,,\; x_2=\frac{1-\sqrt{3}\,i}{4}\,,\; x_3=\frac{2+\sqrt{3}}{2}\,,\; x_4=\frac{2-\sqrt{3}}{2}\,.$$

(32) implies the desired result for (31) immediately.

Summary. We present and prove explicit formulas for the computation of the sidelengths and area of the terms of the sequence $\{T_n\}_{n=0}^{\infty}$ of outer Vecten triangles with respect to a given initial triangle $A_0B_0C_0$ with sidelengths a_0, b_0, c_0 and area $(A_0B_0C_0)$. The formulas for the n-th term of $\{T_n\}_{n=0}^{\infty}$ are only expressed in terms of $n, a_0, b_0, c_0, (A_0B_0C_0)$. Hence, in order to compute $a_n, b_n, c_n, (A_nB_nC_n)$ one need not traverse through the cumbersome procedure of computing the respective magnitudes for all the intermediate terms $A_iB_iC_i$, $i = 1, 2, \ldots, n-1$.

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