Further Sharpening of Two Triangles Euler's Inequality

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Abstract. We establish a sharpened version, with respect to existing results, of Euler's Inequality relating the circumradius and inradius of two triangles respectively.

1. INTRODUCTION. The well known Euler's inequality or, in some references Chapple's inequality [**1, 2**],

$$R \geq 2\, r, \tag{1}$$

relating the radii R, r of the circumscribed and inscribed circles respectively of a triangle ABC has been the subject of many extensions and generalizations [**1, 2, 3, 4, 5**] and references therein. Especially, when (1) is generalized relating the circumradius R of a triangle ABC and the inradius r' of a triangle $A'B'C'$ we take the inequalities [**3, 4, 5**],

$$\frac{R}{r'} \geq \frac{2}{3}\left(\frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\right), \tag{2}$$

$$R\left(\frac{1}{r'} + \frac{1}{R'}\right) \geq \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\ , \tag{3}$$

with a, b, c the sidelengths opposite to vertices A, B, C, a', b', c' the sidelengths opposite to vertices A', B', C', R' the circumradius of the triangle $A'B'C'$. Notice that inequality (3) sharpens inequality (2), [**4, 5**] since,

$$\frac{3}{2}\,\frac{R}{r'} \geq R\left(\frac{1}{r'} + \frac{1}{R'}\right) \geq \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\ .$$

In this note we establish the following version of two triangles Euler's inequality,

$$\frac{3}{2}\,\frac{R}{r'} \geq R\left(\frac{1}{r'} + \frac{1}{R'}\right) \geq \frac{R}{r'}\sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)} \geq \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\ , \tag{4}$$

with $\hat{A}', \hat{B}', \hat{C}'$ the angles of triangle $A'B'C'$. The sum in (4) is taken over all the cyclic permutations of $(\hat{A}', \hat{B}', \hat{C}')$. Inequality (4) sharpens both inequalities (2), (3).

2. PROOF OF THE MAIN RESULT. In order to prove inequality,

$$\frac{R}{r'}\sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)} \geq \frac{a}{a'} + \frac{b}{b'} + \frac{c}{c'}\ , \tag{5}$$

we make use of the well known Klamkin's inequality [**6**],

$$\frac{1}{2}\sum(x\,y)\sqrt{\sum\frac{1}{x\,y}} \geq (x\,\sin\hat{A}+y\,\sin\hat{B}+z\,\sin\hat{C}), \tag{6}$$

with x, y, z positive real numbers, $\hat{A}, \hat{B}, \hat{C}$ the angles of the triangle ABC. The sums in (6) are taken over all the cyclic permutations of (x, y, z).

Let $(A'B'C')$ be the area, s' the semiperimeter, $h_{a'}, h_{b'}, h_{c'}$ the heights corresponding to vertices A', B', C' of triangle $A'B'C'$ respectively. Substituting $x = h_{a'}$, $y = h_{b'}$, $z = h_{c'}$, $\sin\hat{A} = \frac{a}{2\,R}$, $\sin\hat{B} = \frac{b}{2\,R}$, $\sin\hat{C} = \frac{c}{2\,R}$ in (6) we take,

$$\frac{1}{2}\sum(h_{a'}\,h_{b'})\sqrt{\sum\frac{1}{h_{a'}\,h_{b'}}} \geq \left(h_{a'}\,\frac{a}{2\,R}+h_{b'}\,\frac{b}{2\,R}+h_{c'}\,\frac{c}{2\,R}\right) \Rightarrow$$

$$\frac{1}{2}\sum\left(\frac{4\,(A'B'C')^2}{a'b'}\right)\sqrt{\sum\frac{a'\,b'}{4\,(A'B'C')^2}} \geq \frac{2\,(A'B'C')}{2\,R}\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right) \Rightarrow$$

$$\left(\sum\frac{1}{a'b'}\right)\sqrt{\sum(a'\,b')} \geq \frac{1}{R}\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right) \Rightarrow$$

$$\frac{a'+b'+c'}{a'\,b'\,c'}\sqrt{\sum(a'\,b')} \geq \frac{1}{R}\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right) \Rightarrow$$

$$\frac{2\,s'}{4\,R'\,(A'B'C')}\sqrt{\sum 4\,(R')^2\,\sin\hat{A}'\,\sin\hat{B}'} \geq \frac{1}{R}\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right) \Rightarrow$$

$$\frac{4\,R'\,s'}{4\,R'\,s'\,r'}\sqrt{\sum\sin\hat{A}'\,\sin\hat{B}'} \geq \frac{1}{R}\left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right) \Rightarrow$$

$$\frac{R}{r'}\sqrt{\sum\sin\hat{A}'\,\sin\hat{B}'} \geq \left(\frac{a}{a'}+\frac{b}{b'}+\frac{c}{c'}\right).$$

Now we turn to prove that,

$$R\left(\frac{1}{r'}+\frac{1}{R'}\right) \geq \frac{R}{r'}\sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)}. \tag{7}$$

By Gerretsen's inequality $4\,(R')^2+4\,R'\,r'+3\,(r')^2 \geq (s')^2$ we take that,

$$4\,(R')^2+4\,R'\,r'+3\,(r')^2 \geq (s')^2 \Rightarrow 4\,(R'+r')^2 \geq (s')^2+(r')^2+4\,R'\,r' \Rightarrow$$

$$\left(\frac{R'+r'}{R'}\right)^2 \geq \frac{(s')^2+(r')^2+4\,R'\,r'}{4\,(R')^2} = \frac{(s')^2-(s')^2+\sum(a'\,b')}{4\,(R')^2} = \frac{\sum(a'\,b')}{4\,(R')^2} \Rightarrow$$

$$1+\frac{r'}{R'} \geq \sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)} \Rightarrow \frac{R}{r'}\left(1+\frac{r'}{R'}\right) \geq \frac{R}{r'}\sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)} \Rightarrow$$

$$R\left(\frac{1}{r'}+\frac{1}{R'}\right) \geq \frac{R}{r'}\sqrt{\sum\left(\sin\hat{A}'\,\sin\hat{B}'\right)}.$$

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